Asymptotic behaviour and uniform-in-time approximations of nonlocal Fokker-Planck equations

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A nonlocal (fractional) Fokker-Planck

NLFP

$$\partial_t u = \frac{1}{\varepsilon^{2s}}[J_\varepsilon^s * u - u] + \mathrm{div}(xu) := A_\varepsilon^s u + \mathrm{div}(xu) := L_\varepsilon^s u \qquad (1)$$

- $t \geq 0, x \in \mathbb{R}^d, \varepsilon \in (0,1], s \in (0,1]$
- $J^s \in L^1 \cap L^p$ and $J_\varepsilon^s(x) = \varepsilon^{-d} J^s(x/\varepsilon)$
- $\hat{J}^s(\xi) = 1 - |\xi|^{2s} + R_s(\xi)$ where $|R_s(\xi)| \leq C|\xi|^{2s+\delta}$
- As $\varepsilon \to 0$ the operator A_ε^1 approximates the Laplacian Δ and A_ε^s approximates the fractional Laplacian $-(-\Delta^s)$
- Not singular kernel, no regularization.

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Questions

- Does this equation behave like the (fractional) Fokker-Planck for large times?
- Is there a positivity estimate valid as $\varepsilon \to 0$
- Can we show exponential convergence towards the equilibrium **uniformly** in ε e s? What's the shape of this equilibrium?
- Can we estimate the speed of convergence of $e^{L^s_\varepsilon t} u_0$ to $e^{L^s_0 t} u_0$ **uniformly in time**? What about the limit for $s \to 1^-$?

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Motivation

- Models arising in biology e.g., genetic circuits **[Cañizo, Carrillo, Pajaro, 2019]**, growth fragmentation **[Caceres, Cañizo, Mischler, 2011]**: entropy methods
- The latter are not easy to employ in the ε-scaling.
- Harris's Theorem to get the correct behaviour as $\varepsilon \to 0$. Toy model.
- Numerical methods: preservation of the the long time behaviour of its limiting equation (**[Ayi, Herda, Hivert, Tristani, 2022]**, **[Dujardin, Herau, Lafitte, 2020]** etc..)
- Links with (generalized) Central Limit Theorem.

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Previous results

- Nonlocal Diffusion **[Andreu et al., 2010]**: for every $T > 0$

$$\lim_{\varepsilon \to 0} \left\| e^{A_\varepsilon^s t} u_0 - e^{-(-\Delta^s)t} u_0 \right\|_{L^\infty(\mathbb{R}^d \times (0,T))} = 0$$

- Nonlocal Diffusion **[Rey & Toscani, 2012]**: $s = 1$ asymptotic with speed of convergence in Fourier distance
- Nonlocal Fokker Planck **[Mischler & Tristani, 2017]**: different hypotheses on J and on the weights; splitting of the operator.
- Related equations: e.g. **[Ignat & Rossi, 2007]**, **[Molino & Rossi, 2019]**, **[Auricchio, Toscani, Zanella, 2023]**.
- Others...

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Main result

Theorem (Cañizo, T. (2024))

Under suitable hypotheses on J, *and* k, *there exists a unique equilibrium* $F^s_\varepsilon \in L^1_k$ *of equation* (1) *such that for* $u_0 \in L^1_k$,

$$\|u(t,\cdot) - F^s_\varepsilon\|_{L^1_k} \leq Ce^{-\lambda t}\,\|u_0 - F^s_\varepsilon\|_{L^1_k} \qquad \text{for every } t \geq 0. \tag{2}$$

with $C \geq 1$ *and* $\lambda > 0$ **independent** *of* ε *and* s.

- The method is constructive and the constants are explicit
- $-\lambda$ is not the first eigenvalue but it provides a bound of it.

Harris's Theorem

Let S_t be a stochastic semigroup on $\mathcal{M}_V$

1. Confining Lyapunov condition: there exist $T > 0$, $0 < \lambda_L < 1$, and $K > 0$, such that

$$\|S_T \mu\|_V \leq (1 - \lambda_L) \|\mu\|_V + K \|\mu\|$$

2. A uniform positivity condition on a set $\mathcal{C}$: there exist $T > 0$, $0 < \alpha < 1$ and a probability η such that

$$S_T \mu \geq \alpha \eta \int_{\mathcal{C}} \mu$$

Harris's Theorem

If a semigroup $(S_t)_{t \geq 0}$ satisfies the previous two hypotheses with $\mathcal{C}$ "big enough", then the semigroup has a unique invariant probability measure $\mu^* \in \mathcal{P}_V$ and there exist $\lambda > 0$, $C \geq 1$ such that

$$||S_t \mu - \mu^*||_V \leq C e^{-\lambda t} ||\mu - \mu^*||_V \qquad \text{for } t \geq 0$$

Plan

We want to use Harris's Theorem[1]

We have to prove the two conditions:

- Lyapunov condition is fairly straightforward
- Positivity
  - No regularization effect
  - Easy for a fixed ε, while much harder to obtain it uniformly in ε: $\lambda_\varepsilon \to 0$!
  - Uniform in s?

[1] Harris(1956); Meyn & Tweedie (1992, 1993); Hairer & Mattingly (2011); Cañizo & Mischler(2023)

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Positivity

- Write the solution u via Wild sums after some manipulation

$$u(t,x) = e^{(d-\frac{1}{\varepsilon^{2s}})t} u_0(e^t x) + e^{(d-\frac{1}{\varepsilon^{2s}})t} \sum_{n=1}^{\infty} \left(\frac{1}{\varepsilon^{2s}}\right)^n \int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_1} J_\varepsilon^{t_1,\dots,t_n} * u_0(e^t x) dt_1 \dots dt_n.$$

where $J_\varepsilon^{t_1,\dots,t_n}(x) := J^s_{\varepsilon e^{t_1}} * \cdots * J^s_{\varepsilon e^{t_n}}(x)$.

- We want to bound the latter, independently on ε and n.
- L^∞ Berry-Esseen Generalized Central Limit Theorem

Berry-Esseen Theorem

Theorem

Let $f \in \mathcal{P}(\mathbb{R}^d) \cap L^p(\mathbb{R})$ *such that*

$$\hat{f}(\xi) = 1 - |\xi|^{2s} + O(|\xi|^{2s+\delta})$$

and define

$$\mathfrak{f}_n(x) := (\bar{\sigma}_n)^d f_{\sigma_1} * f_{\sigma_2} * \cdots * f_{\sigma_n}(\bar{\sigma}_n x),$$

with $\bar{\sigma}_n^{2s} = \sum_{i=1}^n \sigma_i^{2s}$. *Then, there exist* $N > 0$ *and a constant* C_{BE} *such that for all* $n \geq N$

$$\|\mathfrak{f}_n - G^s\|_{L^\infty} \leq \frac{C_{BE}}{n^{\delta/2s}}.$$

Idea of the proof: high and low frequencies **[Goudon, Junca, Toscani, 2002]** **[Hauray & Mischler, 2014]**

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Positivity, again

There exists an explicit $\varepsilon_0(N)$ such that

- For $\varepsilon \in [\varepsilon_0, 1]$ positivity is straightforward
- For $\varepsilon < \varepsilon_0$,
$$J_\varepsilon^{t_1,\dots,t_n}(x) \geq A \qquad \text{for all } x \in B_\eta$$
for all $\varepsilon < \varepsilon_0$ and for any $t_1, \dots, t_n$ with $t \geq t_1 \geq \dots t_n \geq 0$ and n such that
$$\frac{t}{\varepsilon^{2s}} \leq n \leq 2\frac{t}{\varepsilon^{2s}}.$$
- Then, formally

$$u(t,x) \geq e^{(d-\frac{1}{\varepsilon^{2s}})t} \sum_{n=\frac{t}{\varepsilon^{2s}}}^{2\frac{t}{\varepsilon^{2s}}} \varepsilon^{-2sn} \int_0^t \cdots \int_0^{t_{n-1}} \int_{B_{R_2}} J_\varepsilon^{t_1,\dots,t_n}(e^t x - y) u_0(y)\,\mathrm{d}y\,\mathrm{d}t_n \dots \mathrm{d}t_1$$

$$\geq A e^{dt} e^{-\frac{t}{\varepsilon^{2s}}} \sum_{n=\frac{t}{\varepsilon^{2s}}}^{2\frac{t}{\varepsilon^{2s}}} \left(\frac{t}{\varepsilon^{2s}}\right)^n \frac{1}{n!} \int_{B_{R_2}} u_0(y)\,\mathrm{d}y \geq A C_L e^{dt} \int_{B_{R_2}} u_0(y)\,\mathrm{d}y$$

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Convergence: nonlocal to local for $s = 1$

- With additional assumptions on J, for a nice fast enough decaying φ, we prove the consistency of the operator L_ε,

$$\|(L_\varepsilon - L_0)\varphi\|_{L^1_k} \leq C\varepsilon$$

- Consistency + Hille Yosida ($\left\| L_\varepsilon^{-1} \right\| \leq \frac{C}{\lambda}$) give the speed of convergence of the equilibrium towards the standard Gaussian

$$\|F_\varepsilon - G\|_{L^1_k} \leq C\varepsilon$$

Theorem (Nonlocal to local)

Under additionally regularity and decaying assumptions on u_0, for every $t \geq 0$ and $\varepsilon \in (0, 1]$

$$\left\| e^{L_\varepsilon t} u_0 - e^{L_0 t} u_0 \right\|_{L^1_k} \leq C\varepsilon$$

- Consistency + stability give convergence for finite time
- "Spectral gap" + convergence of the equilibrium

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Convergence: nonsingular to singular for $s \in [s_0, 1)$

Assume that δ is such that there exists $\delta' > 0$ such that $2s_0 + \delta = 2 + \delta'$. We can prove

- Consistency: For all $s \in [s_0, 1)$

$$\|L_\varepsilon^s \varphi - L_0^s \varphi\|_{L_k^1} \leq C\varepsilon^\gamma$$

  where γ depends on δ' but NOT on s
- If one does not pay attention, the rate of convergence goes like $\varepsilon^{(1-s)}$
- Proceeding as before we can prove

Theorem (Nonsingular to singular)

Under additional regularity and decaying assumptions on u_0, for all $t \geq 0$, $\varepsilon \in (0, 1]$, $s \in [s_0, 1]$

$$\left\| e^{L_\varepsilon^s t} u_0 - e^{L_0^s t} u_0 \right\|_{L_k^1} \leq C\varepsilon^\gamma$$

- γ is explicit but not optimal since it comes from some interpolation inequality

Anomalous diffusion to classical diffusion

Assume that δ is such that there exists $\delta' > 0$ such that $2s_0 + \delta = 2 + \delta'$, and assume that J^s converges to J^1 in the following sense^s

$$\left|\hat{R}_s(\xi) - \hat{R}_1(\xi)\right| \leq C(1-s)^{\alpha}|\xi|^{2+\delta'}$$

Theorem

Under additional regularity and decaying assumption on u_0, for all $t \geq 0$, $\varepsilon \in (0,1]$, $s \in [s_0, 1]$,

$$\left\| e^{L^s_\varepsilon t} u_0 - e^{L_\varepsilon t} u_0 \right\|_{L^1_k} \leq C_1 (1-s)^{\beta}(1 + C_2 \varepsilon^{\gamma})$$

Again β, γ are explicit but not optimal

- Example: stable laws

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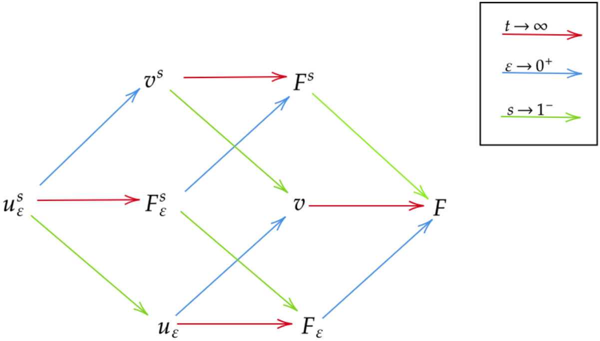
In short...

We are investigating the following limits



v and v^s are the solution of the classical and fractional Fokker Planck.

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Generalizations

- Nonlocal Fokker-Planck with a different potential
- Nonlocal Kinetic Fokker-Planck and linear BGK
- Nonautonomous equations: what if $\varepsilon = \varepsilon(t)$?

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- Let's go back to the pure nonlocal (fractional) diffusion equation

$$\partial_\tau w = J * w - w$$

- Perform a self-similar type change of variables

$$u(t,x) = e^{dt} w\left(\frac{1}{2s}(e^{2st}-1), e^{t}x\right)$$

and obtain the (non-autonomous) nonlocal Fokker-Planck equation

$$\partial_t u = e^{2st}(J_{e^{-t}} * u - u) + \mathrm{div}(xu)$$

If we fix $\varepsilon = e^{-t}$ we obtain again (1)

[2] Ongoing work with José Cañizo (IMAG) and Stéphane Mischler (Université Paris Dauphine)

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Nonautonomous nonlocal equations

Growth-Fragmentation

- Consider the following growth-fragmentation equation **[Perthame, 2007]**

$$\partial_t f(t,x) = \int_x^\infty K(y,x) f(t,y)\,\mathrm{d}y - \int_0^x \frac{y}{x} K(x,y) f(t,x)\,\mathrm{d}y = \mathcal{L}^+ f - Bf$$

- Self-similar change of variables

$$g(t,x) = e^{-2t} f(e^{\gamma t} - 1, e^{-t} x)$$

  obtaining

$$\partial_t g + g + \partial_x(xg) = \gamma \mathcal{L}^+_{e^{-t}} g - \gamma B_{e^{-t}} g$$

  with kernel $K_\varepsilon(x,y) := \varepsilon^{-(\gamma-1)} K(\varepsilon x, \varepsilon y)$.

- In **[Cáceres, Cañizo, Mischler (2011)]** assuming homogeneity

$$K(rx, ry) = r^{\gamma-1} K(x,y),$$

  asymptotic convergence was established using entropy methods.

- Drop the homogeneity condition and work with the nonautonomous equation instead:
  Harris' theorem for two-parameter semigroups.

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THANK YOU!

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